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On causality and closed geodesics of compact Lorentzian manifolds and static spacetimes

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Abstract

Some results related to the causality of compact Lorentzian manifolds are proven: (1) any compact Lorentzian manifold which admits a timelike conformal vector field is totally vicious, and (2) a compact Lorentzian manifold covered regularly by a globally hyperbolic spacetime admits a timelike closed geodesic, if some natural topological assumptions (fulfilled, for example, if one of the conjugacy classes of deck transformations containing a closed timelike curve is finite) hold. As a consequence, any compact Lorentzian manifold conformal to a static spacetime is geodesically connected by causal geodesics, and admits a timelike closed geodesic.

MSC: 53C50; 53C22; 58E10

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1. Introduction

The aim of this paper is to give some general results on causality and existence of closed timelike geodesics in compact Lorentzian manifolds (Theorems 1.1, 1.2, Propositions 4.2, 4.4). Even though these

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results are interesting in themselves, the principal application will hold for compact static spacetimes (Theorem 1.3).[1]

A well-known result by Tipler [32] (see also [4, Theorem 4.15]) asserts that any compact Lorentzian manifold, covered regularly by a globally hyperbolic manifold which admits a *compact* Cauchy hypersurface, must contain a closed timelike geodesic. This result was extended by Galloway [10], who also introduced the notion of *stable* free t-homotopy class (see also [11] for related results). Recently, Guediri [13] has shown that the hypothesis on compactness in Tipler's result cannot be removed, by means of a counterexample (see also [7,14,15]). Nevertheless, the compactness hypothesis can be replaced by the following assumption [13, Theorem 5.1]: a free t-homotopy class is determined by a central deck transformation ϕ, i.e., ϕ is the unique element in its conjugacy class $\mathcal{C}$. Later on, Caponio et al. [6] have studied compact static spacetimes by using some variational results. Essentially, they show that such a spacetime is geodesically connected and, if a free homotopy class is determined by a finite conjugacy class of deck transformations $\mathcal{C}$, it contains a closed geodesic (not necessarily timelike).

The results in the present article can be summarized as follows.

In Section 2 some preliminary properties are recalled, with particular emphasis in the geometrical and topological properties of static spacetimes, which will be necessary to apply our results. Especially, Theorem 2.1 and Corollary 2.3 characterize when the universal covering of a static spacetime is standard static, and its main properties.

In Section 3 we prove the following result on the causality of a class of Lorentzian manifolds. Recall that a (time-oriented) Lorentzian manifold (M,g) is called totally vicious if the chronological future and past of any point is the whole manifold, i.e., $I^+(p) = I^-(p) = M, \forall p \in M$.

Theorem 1.1. *Any compact Lorentzian manifold (M,g) which admits a timelike conformal vector field is totally vicious.*

The technique of the proof involves some properties of conformal vector fields studied in [22]. Theorem 1.1 will be essential to prove not only that a compact static manifold is geodesically connected, but also that any two points can be joined by a timelike geodesic (see Theorem 1.3(1)).

In Section 4, we give two extensions of Tipler's result, where the compactness of the Cauchy hypersurface is replaced by different assumptions on the group of deck transformations, Propositions 4.2, 4.4. As an immediate consequence of the first one (Proposition 4.2), we have the following generalization of Guediri's criterion for the existence of timelike closed geodesics [13, Section 5]:

Theorem 1.2. *Let (M,g) be a compact Lorentzian manifold which admits a regular covering $\Pi : \bar{M} \to M$ such that $\bar{M}$ is globally hyperbolic, and let G be the group of deck transformations of $\bar{M}$. Assume that a conjugacy class $\mathcal{C} \subset G$ satisfies*:

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[1] Any compact spacetime does contain closed timelike curves (CTCs). From a classical relativistic viewpoint, the existence of CTCs is a drawback for a spacetime because of well-known paradoxes [17, p. 189]. Nevertheless, for different reasons there has been a continued interest in spacetimes with CTCs: the existence of CTCs in classical spacetimes such as Gödel's or the inner part of Kerr's, technical advantages of compactifications, speculations on time-machines and wormholes, quantum interpretations, the recent role of the Gödel solution as an exact model in string theory (see for example [8,18,33,34])... Nevertheless, our study will remain at a geometrical level.

(a) *It contains a closed timelike curve* α.
(b) $\mathcal{C}$ *is finite.*

Then there exists at least one closed timelike geodesic in $\mathcal{C}$.

As a consequence of the second extension (Proposition 4.4), in Section 5 the results on closed geodesics in [6] will be improved by showing that, in a compact static spacetime, a closed *timelike* geodesic exists, *without any further assumption on the fundamental group*. Moreover, our results will also hold under conformal transformations, because are based only on causal and topological properties (in the spirit of [28]). Summing up, we will prove and discuss:

Theorem 1.3. *Let* (M, g) *be a compact static spacetime. Then*:

(1) *Any pair of points* $p, q \in M$ *can be joined by means of a timelike geodesic.*

(2) *Any conjugacy class* $\mathcal{C} \subset G$ *which contains a closed timelike curve contains a closed timelike geodesic too. In particular, there exists at least one closed timelike geodesic in* M.

2. Preliminaries. Static spacetimes

All Lorentzian manifolds are assumed to be connected, time-oriented (thus, time-orientable), with dimension $n \geqslant 2$. As usual, differentiability C^∞ will be assumed, even though, in principle, we only need C^1 (geodesics and causality are then well-defined). Our notation and conventions will be standard in Lorentzian Geometry, as in the books [4,21,24]. A Lorentzian manifold will be called *stationary* if it admits a timelike Killing vector field K, and *static* if, additionally, K is *irrotational* (the orthogonal distribution to K is involutive). Standard properties of such manifolds can be seen in [24], and a survey in [30]; for recent references on the static case see[2] [1,6,27].

The problem of the geodesic connectedness of a Lorentzian manifold has been widely studied recently, specially since Masiello's book [19], which develops a variational viewpoint (see [26] for a survey). Nevertheless, our results on connectedness will rely on a classical theorem by Avez [2] and Seifert [31] for causal geodesics: *in any globally hyperbolic spacetime, each two causally related points* p, q *can be joined by a causal geodesic, with length equal to the time-separation* (*or Lorentzian distance*) *between* p *and* q.

If the Lorentzian manifold (M, g) is compact, it is well known that the Euler characteristic of M vanishes. Even though this is not a restriction if the dimension n is odd, it yields a first topological restriction for even n; in particular, if $n = 2, 4$ then M cannot be simply connected. The condition of stationarity yields new topological obstructions [22]; for example, if (M, g) is compact, stationary and $n = 3$ then M is a Seifert manifold. Nevertheless, by using Hopf fibration it is not difficult to construct stationary metrics on any odd-dimensional sphere [23] (see [16] for further properties). Thus, taking the product of such a stationary 3-sphere by any Riemannian k-sphere, with $k > 1$, we have: *there exist simply connected compact stationary manifolds of any dimension* $n \geqslant 5$ *and* $n = 3$.

The situation is radically different in the static case. In fact, if (M, g) is a static manifold and K is the corresponding "static" (irrotational timelike Killing) vector field, then K is parallel for the conformal

[2] A brief survey of the static case is carried out in [25], including an announcement of the results in the present article.

metric

$$g^* = -\frac{1}{g(K,K)} g \tag{2.1}$$

(notice that any static spacetime is locally isometric to a standard one $\mathbb{R} \times S$ endowed with a metric as (2.3) below, with K identifiable to ∂_t). Thus, the associated one-form

$$\omega = -g^*(K, \cdot) \tag{2.2}$$

is closed and, if M is compact, then it cannot be simply connected. Moreover, the following structural result holds (compare with [6, Section 3]):

Theorem 2.1. *Let (M, g) be a static manifold with static vector field K, and $(\bar{M}, \bar{g})$, $\Pi : \bar{M} \to M$, $\bar{g} = \Pi^* g$, its universal Lorentzian covering.*

(1) *If the vector field K is complete then $(\bar{M}, \bar{g})$ is a standard static manifold. More precisely, $\bar{M}$ is isometric to a product $\mathbb{R} \times S$ endowed with the metric*

$$\bar{g}\big[(t,x)\big] = -\beta(x)\, dt^2 + g_S[x] \tag{2.3}$$

where g_S is a Riemannian metric on S, $dt = \Pi^*\omega$, $K_{\Pi(p)} = d\Pi_p(\partial|_p)$ and, being $\Pi_S : \mathbb{R} \times S \to S$ the natural projection, $\beta(\Pi_S(p)) = -g(K_{\Pi(p)}, K_{\Pi(p)})$, for all $p \in \bar{M}$.

(2) *If the metric g is (geodesically) complete, then K is complete and the metric g_S in (2.3) is complete.*

Proof. (1) Let $\bar{K}$ be the (complete) vector field on $\bar{M}$ such that $\Pi_* \bar{K} = K$, and let $\bar{\Phi}$ be its global flow. As $\bar{M}$ is simply connected, the closed form $\Pi^*\omega$ is exact, i.e., $\Pi^*\omega = dt$ for some function $t : \bar{M} \to \mathbb{R}$. Fixing $p \in \bar{M}$ one has $t(\bar{\Phi}_s(p)) = s + t(p)$ for all $s \in \mathbb{R}$ (use $dt(K) \equiv 1$). Thus, putting $S = t^{-1}(0)$, it is straightforward to check that the required isometry is:

$$\bar{M} \to \mathbb{R} \times S, \qquad p \mapsto \big(t(p), \bar{\Phi}_{-t(p)}(p)\big).$$

(2) Let us see that the vector field K must be complete. Otherwise, for some $p \in M$, a local flow Φ of K will satisfy that the curve $\lambda \to \Phi_\lambda(p)$ is well defined for $\lambda \in [0, 1)$ but cannot be continuously extended to $\lambda = 1$. By using the local decomposition of M as a standard static spacetime, there exists a neighborhood U of p isometric to $(-\nu, \nu) \times S_p$, for some $\nu > 0$, endowed with a metric as (2.3). Now, for a small $\mu > 0$ ($\mu < \nu \leqslant 1$) there exists a geodesic

$$\gamma : [0,1] \to U \equiv (-\nu, \nu) \times S_p \quad \text{with } \gamma(0) = p\big(\equiv (0,p)\big),\ \gamma(1) = \Phi_\mu(p)\big(\equiv (\mu, p)\big).$$

For each $\lambda \in [0, 1-\mu]$, consider the (*complete*) geodesic γ_λ with initial condition:

$$\gamma_\lambda(0) = \Phi_\lambda(p) \quad \text{and} \quad \gamma'_\lambda(0) = d\Phi_\lambda\big(\gamma'(0)\big).$$

Clearly, for some $\lambda_0 > 0$ one has:

$$\gamma_\lambda(s) = \Phi_\lambda \circ \gamma(s), \quad \forall s \in [0,1],\ \forall \lambda \in [0, \lambda_0]. \tag{2.4}$$

Assume that the domain V_λ of Φ_λ ($\Phi_\lambda : V_\lambda \to M$) is a maximal neighbourhood of p, and define an interval $I \subseteq [0, 1-\mu]$ by: $\lambda_0 \in [0, 1-\mu]$ belongs to I if and only if (2.4) holds. The result will hold if $I = [0, 1-\mu]$ because, in this case,

$$\gamma_{1-\mu}(1) = \Phi_{1-\mu}\big(\gamma(1)\big) = \Phi_1(p),$$

in contradiction with the inextendibility of $\Phi_\lambda(p)$. As $\gamma([0,1])$ is compact, the interval I is an open subset of $[0, 1-\mu]$. Therefore, if $I \neq [0, 1-\mu]$, then $I = [0, \lambda_{\max})$ for some $0 < \lambda_{\max} \leqslant 1$, and the following contradiction would appear. Taking into account that the limit of $\gamma'_\lambda(0)$ when $\lambda \nearrow \lambda_{\max}$ is $\gamma'_{\lambda_{\max}}(0)$, one has

$$\gamma_{\lambda_{\max}}(s) = \lim_{\lambda \nearrow \lambda_{\max}} \Phi_\lambda \circ \gamma(s), \qquad \forall s \in [0,1]$$

and each integral curve $\lambda \to \Phi_\lambda(\gamma(s))$ can be continuously extended beyond $\lambda_{\max}$, that is, the maximal domain $V_{\lambda_{\max}}$ of $\Phi_{\lambda_{\max}}$ contains $\gamma(s)$ for all $s \in [0,1]$, and $\lambda_{\max} \in I$.

For the last assertion, recall that any maximal integral manifold S of the kernel of ω will be complete, because S is totally geodesic in M. □

Remark 2.2. (1) If M were compact then not only K would be complete but the static metric g would be complete too (see [22]); thus, Theorem 2.1(2) would be applicable.

(2) A static standard manifold $(\bar{M}, \bar{g})$ as in (2.3) is globally hyperbolic if g_S is complete and β behaves at most quadratically at infinity. (Recall the definition: let d_R be the distance on S canonically associated to the Riemannian metric g_S, and assume that, for some fixed $x_0 \in S$ and $k, k' \in \mathbb{R}$, $p > 0$,

$$\beta(x) \leqslant k d_R^p(x, x_0) + k'; \tag{2.5}$$

if (2.5) holds for $p = 2$ (resp. some $p < 2$) then β is said to behave *at most quadratically* (resp. *sub-quadratically*) at infinity.) In fact, the conformal metric $g_S^* = g_S/\beta$ would be complete too and, thus, $\bar{g}^* = \Pi^*(g^*)$ would be globally hyperbolic, each slice $\{t_0\} \times S$ being a spacelike Cauchy hypersurface[3] [4, Theorem 3.67]. As $\bar{g}$ is globally conformal to $\bar{g}^*$, $(\bar{M}, \bar{g})$ is globally hyperbolic too. In particular, this happens for $(\bar{M}, \bar{g})$ in Theorem 2.1 if M is compact, because g would be complete and $\mathrm{Sup}(\beta) < \infty$.

(3) As dt is the pull-back of ω, any deck transformation ϕ of $\bar{M}$ must preserve dt (i.e., $dt = \phi^* dt = d(t \circ \phi)$), and $t \circ \phi = t + T_\phi$, for some $T_\phi \in \mathbb{R}$, i.e.:

$$\phi(t, x) = \big(t + T_\phi, \Pi_S\big(\phi(t,x)\big)\big), \qquad \forall (t,x) \in \mathbb{R} \times S.$$

By deriving partially in both sides with respect to t, and taking into account that $\phi_*(\partial_t) = \partial_t$, we can write: $\Pi_S(\phi(t,x)) = \phi^S(x)$ (independent of t) for some diffeomorphism ϕ^S of S.

Summing up for the compact case:

Corollary 2.3. *Let (M, g) be a compact static manifold, and $(\bar{M}, \bar{g})$ its universal Lorentzian covering. Then*:

(1) *$(\bar{M}, \bar{g})$ is isometric to a globally hyperbolic standard static spacetime $\mathbb{R} \times S$ as in (2.3), being each slice $\{t_0\} \times S$ a Cauchy hypersurface.*
(2) *Any deck transformation $\phi : \bar{M} \to \bar{M}$ can be written as*

$$\phi(t, x) = \big(t + T_\phi, \phi^S(x)\big),$$

for some diffeomorphism ϕ^S of S and $T_\phi \in \mathbb{R}$.

[3] Even though a (*smooth*) *spacelike* Cauchy hypersurface exists in any globally hyperbolic spacetime, this is not as trivial as it sounds [5].

Example 2.4. Notice that S is not necessarily compact. It is especially easy to construct examples in a torus (recall that any stationary surface is static, because the orthogonal distribution to the timelike Killing vector field K is 1-dimensional; more general examples can be constructed obviously by taking this surface as the fiber of a warped product- or as the base, provided that the warping function is invariant by the flow of K). In fact, it is trivial that any flat Lorentzian torus admits a Killing (indeed, parallel) timelike vector field K, such that the integral curves of $K^\perp$ (which are isometric to S) are not closed. Of course, in this example there are other K's where the curves are closed. But one can also construct a stationary torus with only one independent Killing vector field K such that the integral curves of $K^\perp$ are not closed, as follows. Consider $\mathbb{R}^2$, endowed with the Lorentzian metric

$$g = F(x)(dx \otimes dy + dy \otimes dx) - G(x)\,dy^2 \quad \big(F(x) \neq 0,\ \forall x \in \mathbb{R}\big),$$

where F, G are periodic functions of period 1, and let T^2 be the Lorentzian torus obtained as the quotient $\mathbb{R}^2/\mathbb{Z}^2$ (these metrics, as well as those in Remark 3.2 below, are particular cases of Lorentzian tori admitting a Killing vector field, studied systematically in [29]). The Killing vector field ∂_y projects onto a Killing vector field K on T^2. If $G > 0$, K is timelike, and T^2 is static. By [29, Theorem 4.2], if $G' \not\equiv 0$, the metric is not flat and any other Killing vector field on T^2 is a multiple of K. Now, recall that the vector field $G(x)\partial_x + F(x)\partial_y$ on $\mathbb{R}^2$, projects onto a non-vanishing vector field W on T^2 orthogonal to K. Finally, it is easy to check that if

$$\int_0^1 \frac{F}{G}(x)\,dx$$

is not rational, then the integral curves of W are not closed, as required.

Remark 2.5. Some additional information on (M, g) in Corollary 2.3 can be obtained, in comparison with the general results in [35]. Recall that the Levi-Civita connection ∇^* of g^* in (2.1) is Riemannian, i.e., the Riemannian metric $g^*_R(A, B) = g^*(A, B) - 2g^*(A, K)g^*(B, K)$ has the same Levi-Civita connection that ∇^*. Then, deck transformations for $\bar{M}$ are also isometries for both, $\bar{g}^* = \Pi^*(g^*)$ and $\bar{g}^*_R = \Pi^*(g^*_R)$. Now, write $(\bar{M}, \bar{g}^*)$ as a semi-Riemannian product $\mathbb{L}^k \times N$ where $\mathbb{L}^k, k \geqslant 1$, is a k-dimensional Lorentz Minkowski spacetime (∂_t will be chosen to project on K), and N is a Riemannian manifold with no further decomposition as a Riemannian product ($N \neq N' \times \mathbb{R}$). Thus, any deck transformation ϕ of $\bar{M}$ can be written as a composition $\phi_1 \circ \phi_2$, where ϕ_1 is an isometry of N and ϕ_2 is an isometry of $\mathbb{L}^k$ (and $\mathbb{R}^k$) which preserves ∂_t (i.e., ϕ_2 can be identified to an element of the semi-direct product $O(k-1, \mathbb{R}) \times \mathbb{R}^k = (O(k, \mathbb{R}) \cap O_1^\uparrow(k, \mathbb{R})) \times \mathbb{R}^k$).

3. Connecting timelike curves

Notice that a timelike conformal vector K for g is Killing and unitary for the conformal metric $g^* = -(1/g(K, K))g$ (see [29, Lemma 2.1]). Thus, Theorem 1.1 is equivalent to:

Proposition 3.1. *Any compact Lorentzian manifold (M, g) admitting a Killing vector field K with $g(K, K) = -1$ is totally vicious.*

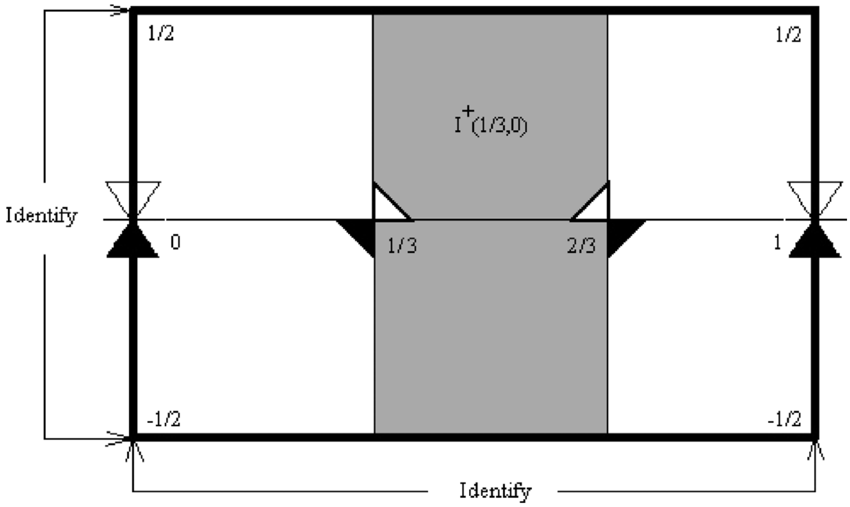

Fig. 1.

Proof. It is not difficult to prove that a Lorentzian manifold is totally vicious if and only if for every point p there exists a closed timelike curve through p (see [20, Proposition 2.2]). Thus, it is enough to show that (M, g) admits a timelike vector field $X \in \Gamma(TM)$ with closed integral curves. Consider the auxiliary Riemannian metric $g_R(A, B) = g(A, B) - 2g(A, K)g(B, K)$ for all $A, B \in \Gamma(TM)$, which will have a compact isometry group $\mathrm{Iso}(M, g_R)$. A straightforward computation shows that K is also a Killing vector field for g_R and, thus, its one-parameter group G has a compact closure $\bar{G}$ in $\mathrm{Iso}(M, g_R)$. As G is abelian, $\bar{G}$ is abelian too and, thus, isomorphic to a k-torus, for some $k \geqslant 1$. Therefore, there is a sequence of one-parameter subgroups $\{G_m\}$ diffeomorphic to circles, whose associated sequence of g_R-Killing vector fields $\{X_m\}$ converges to K (i.e., $\lim_{m\to\infty} \mathrm{Max}_{p\in M}\, g_R(X_m(p) - K(p), X_m(p) - K(p)) = 0$). Thus, for some m_0 sufficiently large, X_{m_0} is timelike, and we can choose $X = X_{m_0}$. $\square$

Remark 3.2. Total viciousness may *not* hold if the compact Lorentzian manifold (M, g) is assumed to admit a Killing vector field K which is only *causal*. In fact, it is not difficult to construct counterexamples among Lorentzian tori $\mathbb{R}^2/\mathbb{Z}^2$ obtained as a quotient of $\mathbb{R}^2$ endowed with the metric $g = \sin(\psi(x))(dx^2 - dy^2) + 2\cos(\psi(x))\,dx\,dy$ for suitable functions $\psi(x)$ of period 1 (see Fig. 1). Recall that, in these counterexamples, $K = \partial_y$ is also irrotational.

4. Closed timelike geodesics

Let (M, g) be a compact Lorentzian manifold and $\Pi : \bar{M} \to M$ a regular covering endowed with the pullback metric $\bar{g} = \Pi^* g$. Assume that $(\bar{M}, \bar{g})$ is globally hyperbolic and, thus, topologically $\bar{M} = \mathbb{R} \times S$, where S is a Cauchy hypersurface. Let $d : \bar{M} \times \bar{M} \to [0, \infty)$ be the Lorentzian time-separation (or Lorentzian distance) on $\bar{M}$. Recall that, because of global hyperbolicity, d is continuous and finite-valued. For each deck transformation $\phi \in G$, consider the function

$$d_\phi : \bar{M} \to [0, \infty), \quad d_\phi(p) = d\big(p, \phi(p)\big), \quad \forall p \in \bar{M}. \tag{4.1}$$

Lemma 4.1. *If d_ϕ restricted to S admits a relative maximum p_0 with $d_\phi(p_0) > 0$ then there exists a timelike closed geodesic in (M, g).*

Proof. The result is a consequence of Tipler's technique. In fact, as $\bar{M}$ is globally hyperbolic and $d_\phi(p_0) > 0$ (i.e., $\phi(p_0) \in I^+(p_0)$) the Avez–Seifert result yields a timelike geodesic $\bar{\gamma}: [0, 1] \to \bar{M}$ from p_0 to $\phi(p_0)$ which is maximizing, i.e., length$(\bar{\gamma}) = d_\phi(p_0)$. Then, $\gamma = \Pi \circ \bar{\gamma}$ is the required geodesic (otherwise, $\gamma'(0) \neq \gamma'(1)$ and γ could be modified in any arbitrarily smooth neighborhood of $\gamma(0)(= \gamma(1))$ to obtain a strictly longer closed timelike curve, which contradicts the condition of relative maximum of p_0). □

Let $\mathcal{C}$ be a conjugacy class of the group G of deck transformations of $\bar{M}$. Even though, in general, a closed curve α does not determine any deck transformation, α does determine a conjugacy class of G and, thus, to assert that $\mathcal{C}$ contains α makes sense. Even more, in the special case of closed timelike curves, if γ_1 and γ_2 are two freely t-homotopic closed curves (in the sense of [10], i.e., freely homotopic through timelike curves) with base points x_1, x_2 on each one, and if $\bar{x}_1$, $\bar{x}_2$ are two points on $\bar{M}$ over x_1, x_2, resp., both belonging to the same Cauchy hypersurface, then γ_1 and γ_2 determine the same deck transformation. In particular, given a free t-homotopy class $\tilde{\mathcal{C}}$, each Cauchy hypersurface determines a unique deck transformation in the conjugacy class representing $\tilde{\mathcal{C}}$.

Proposition 4.2. *Let (M, g) be a compact Lorentzian manifold which admits a globally hyperbolic manifold $(\bar{M}, \bar{g})$ as a regular covering. Assume that a conjugacy class of deck transformations $\mathcal{C} \subset G$ satisfies*:

(a) *$\mathcal{C}$ contains a (future-directed) closed timelike curve α.*
(b) *For some (and then for any) compact subset $K \subset \bar{M}$ such that $M \subset \Pi(K)$ all the restricted functions $d_\phi|_K$, $\phi \in \mathcal{C}$ are null, except for a finite number $\phi_1, \ldots, \phi_j \in \mathcal{C}$.*

Then there exists a closed timelike geodesic in $\mathcal{C}$.

Proof. Fix a Cauchy hypersurface S, and $\phi_0 \in \mathcal{C}$. From Lemma 4.1, it is enough that d_{ϕ_0} attains an absolute maximum on $\bar{M}$ and, thus, on S. Recall first that, for any deck transformation $\psi \in G$:

$$d_{\phi_0}\big(\psi(p)\big) = d\big(\psi(p), \phi_0\big(\psi(p)\big)\big) = d\big(p, \psi^{-1} \circ \phi_0 \circ \psi(p)\big) = d_{\psi^{-1}\phi_0\psi}(p). \tag{4.2}$$

On the other hand, taking into account that $\Pi(K) = M$ and the covering is regular:

$$\big\{d_{\phi_0}(p)\colon\ p \in \bar{M}\big\} = \big\{d_{\phi_0}\big(\psi(p)\big)\colon\ p \in K, \psi \in G\big\}. \tag{4.3}$$

As only finitely many conjugate $\psi^{-1} \circ \phi_0 \circ \psi \in \mathcal{C}$ are non-identically zero on K, the supremum in (4.2) is then equal to the maximum of

$$\big\{d_{\phi_i}(p_i),\ i = 1, \ldots, j\big\},$$

attained at some index i_0, where each p_i is the maximum of the non-null function $d_{\phi_i}|_K$. Therefore, by (4.2) the absolute maximum of d_{ϕ_0} is attained at $\psi_{i_0}(p_{i_0})$, where $\phi_0 = \psi_{i_0} \circ \phi_{i_0} \circ \psi_{i_0}^{-1}$. □

Remark 4.3. Assumption (a) is natural and not too restrictive, because any compact Lorentzian manifold admits a closed timelike curve (see for example [21, Lemma 14.10]). Then, the curve α determines a free t-homotopy class where, in fact, a closed timelike geodesic should appear. Of course, assumption (b) is always satisfied if $\mathcal{C}$ is finite, yielding Theorem 1.2 (compare with Sections 4 and 5 in [13]). Nevertheless, the more general assertion in (b) will be needed to prove Proposition 4.4, as we will see below (see step 2 of the proof of this proposition).

In Lemma 4.1 we saw that the existence of a (relative) maximum of d_ϕ on S was enough to obtain a closed timelike geodesic, but in the proof of Proposition 4.2 we ensured a stronger property, the existence of a maximum on all $\bar{M}$. The reason relies in the lack of good technical properties of $\phi|_S$ when ϕ is changed by other element of $\mathcal{C}$. Nevertheless, there are interesting spacetimes, as the static ones, where these technical properties (namely, (4.4) below) occur. In this case, assumption (b) in Proposition 4.2 can be dropped:

Proposition 4.4. *Let (M, g) be a compact Lorentzian manifold which admits a globally hyperbolic manifold $(\bar{M}, \bar{g})$ as a regular covering. Assume that any deck transformation $\phi \in G$ of $\bar{M} = \mathbb{R} \times S$ can be written as*

$$\phi(t, x) = \big(t + T_\phi, \phi^S(x)\big), \tag{4.4}$$

for some homeomorphism ϕ^S of S and $T_\phi \in \mathbb{R}$.

If the conjugacy class $\mathcal{C}$ contains a closed timelike curve α, then there exists a closed timelike geodesic in $\mathcal{C}$.

Proof. It is straightforward to check that the value of T_ϕ is equal for all the elements in a same conjugacy class; thus, we can write:

$$\phi(t, x) = \big(t + T, \phi^S(x)\big), \quad \forall \phi \in \mathcal{C}. \tag{4.5}$$

Put $S \equiv \{0\} \times S$, let $\phi_0 \in \mathcal{C}$ be a deck transformation determined by α, and let d_S be the restriction of function d_{ϕ_0} to S, i.e.:

$$d_S : S \to [0, \infty), \qquad d_S(x) = d\big((0, x), \big(T, \phi_0^S(x)\big)\big), \quad \forall x \in S. \tag{4.6}$$

From Lemma 4.1, it is enough to show that d_S attains a maximum. The proof consists of the following three steps:

Step 1: To choose a compact subset $K^S \subset S$ such that:

$$S = \bigcup_{\psi \in G} \psi^S(K^S).$$

K^S can be chosen as follows. Consider a compact subset $K \subset \mathbb{R} \times S$ such that $\Pi(K) = M$. Clearly, K can be chosen as a product $K = [0, \bar{T}] \times K^S$, and

$$\bar{M} = \bigcup_{\psi \in G} \psi(K) = \bigcup_{\psi \in G} [T_\psi, T_\psi + \bar{T}] \times \psi^S(K^S).$$

Thus, if $\Pi_S : \mathbb{R} \times S \to S$ is the natural projection:

$$S = \Pi_S(\bar{M}) = \bigcup_{\psi \in G} \psi^S(K^S).$$

Step 2: To show that, fixed T in (4.5), there are only finitely many deck transformations $\phi \in \mathcal{C}$ satisfying

$$\big(T, \phi^S(K^S)\big) \cap J^+(0, K^S) \neq \emptyset \tag{4.7}$$

(i.e., all the d_ϕ's in (4.1) restricted to $(0, K^S)$ are null for $\phi \in \mathcal{C}$, except for a finite subset of ϕ's). Notice first that $(T, S) \cap J^+(0, K^S)$ is compact [28, Lemma 3.1], as well as the following general result: given two compact subsets $K_1, K_2 \subset \bar{M}$ the set

$$\big\{\phi \in \mathcal{C}\colon\ K_1 \cap \phi(K_2) \neq \emptyset\big\}$$

is finite (in fact, the result holds even if ϕ is allowed to vary in all G, see [28, Lemma 3.2]). Then, one has just to apply this result to $K_1 = (T, S) \cap J^+(0, K^S)$ and $K_2 = (0, K^S)$.

Step 3: Reasoning as in the proof of Proposition 4.2, to prove that d_S in (4.6) attains a maximum. In fact, if $\phi_1, \ldots, \phi_j \in \mathcal{C}$ are the only deck transformations such that $\phi_1^S, \ldots, \phi_j^S$ satisfy (4.7), one has:

$$\begin{aligned}\operatorname{Sup}\big\{d_S(p)\colon\ p \in S\big\} &= \operatorname{Sup}\big\{d_{\psi^{-1}\phi_0\psi}(p)\colon\ p \in K^S,\ \psi \in G\big\}\\ &= \operatorname{Max}\big\{d_{\phi_i}(p)\colon\ p \in K^S,\ i = 1, \ldots, j\big\}\end{aligned}$$

(in the first equality step 1 and (4.2) are used, and in the second, step 2). □

5. Application to static spacetimes

Proof of Theorem 1.3. For (1), recall that Theorem 1.1 ensures the existence of a timelike curve α from p to q. Lifting α to a curve $\bar{\alpha}$ in the universal covering $(\bar{M}, \bar{g})$ and using that this is globally hyperbolic (Corollary 2.3), Avez–Seifert result yields a timelike geodesic $\bar{\gamma}$ connecting the endpoints of $\bar{\alpha}$. Thus, the required geodesic is $\gamma = \Pi \circ \bar{\gamma}$.

The part (2) is obvious from Proposition 4.4 and Corollary 2.3, plus Remark 4.3. □

Remark 5.1. (1) We saw in Remark 2.5 that the affine connection ∇^* on M associated to the conformal metric g^* in (2.1) is Riemannian and, thus, g^* is geodesically connected and admits closed geodesics. Nevertheless, this does not imply directly that g also satisfies these two properties because, as far as we know, such properties are not conformally invariant (even on compact manifolds). But, as a clear difference with the techniques in [6], all the properties we have used to prove Theorem 1.3 are explicitly conformally invariant (say, hypotheses as (a), (b) in Proposition 4.2 holds for g if and only if hold for any conformal g^*, because both metrics have equal timelike vectors and relations of causality), and thus:

> If (M, g) is a compact static spacetime and $\Omega : M \to (0, \infty)$ is any function, then the conformal metric $g^* = \Omega g$ also satisfies both conclusions (1) and (2) in Theorem 1.3.

In particular, this is applicable to warped products $(B \times_f F, g = g_B + f^2 g_F)$ where one of the factors, say, the base (B, g_B) is a compact static manifold, and the other (F, g_F) a compact Riemannian manifold. In order to check if a conjugacy class type $\mathcal{C}_B \times \mathcal{C}_F$ contains a closed timelike geodesic: (i) check that $\mathcal{C}_B$ contains closed timelike curves, (ii) compute the maximum L of the lengths of such CTCs for the conformal metric $g_B^* = g_B/f^2$ (this is equal to the g_B^*-length of a maximizing closed timelike g_B^*-geodesic in $\mathcal{C}_B$), (iii) compute the minimum l of the g_F-lengths for curves in $\mathcal{C}_F$ (equal to the length of a minimizing closed g_F-geodesic in F), and (iv) $\mathcal{C}_B \times \mathcal{C}_F$ admits a closed timelike geodesic if and only if $l < L$.

(2) Essentially, Theorem 1.3 improves widely the corresponding results in [6] (for example, Corollaries 4.6, 4.7 and 4.8 in [6] are particular cases). As suggested by the authors of this reference, an interesting open question would be to determine which conclusions of Theorem 1.3 hold if (M, g) is just stationary. Notice that, even though many interesting stationary compact manifolds will satisfy the assumptions of Proposition 4.4 (see for example [28]), there are others which *do not* satisfy them, as the simply connected ones in Section 2. On the other hand, recall that the question whether a compact Lorentzian manifold admits a closed (non-necessarily causal) geodesic [11] remains open, as far as we know.

(3) In the non-compact case, the authors of [6] use the following result (essentially contained in [12], see also [19]): *a standard static spacetime* $\mathbb{R} \times S$ *as in* (2.3) *with* g_S *complete and* β *subquadratic is geodesically connected*. Recall that, in this case, the spacetime is globally hyperbolic too (Remark 2.2(2)). From the results in [9], chosen $\epsilon > 0$, there exist counterexamples to geodesic connectedness even if inequality (2.5) holds with $p = 2 + \epsilon$. Thus, the quadratic case $p = 2$ becomes critical for geodesic connectedness. Nevertheless, even in this case it is possible to prove geodesic connectedness [3].

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